TOWARD A THEORY OF TURBULENCE

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Criteria are established for the transition of laminar flow into turbulent as a consequence of probability estimates of interaction of individual liquid layers.

The well-known phenomenological transport equations (conservation laws) for continuous media can be obtained by use of statistical (kinetic) distribution functions. Single-index (condition-free) correlative functions [1-3] were first used for this purpose, then followed by multindex conditional distribution functions [4]. Basic macrocopic qualities are defined with their aid (mass density, flow velocity, energy density, stress tensor, thermal flux density, etc.).

The averaging operation for any dynamic quantity then involves not only averaging over the molecular momentum space of the medium, but also averaging over a small configuration space volume. Thus, in the conditional distribution it is sufficient to perform averaging over a molecular cell. A consequence of this operation is the "concealment" of the internal microscopic fluctuation mechanism of the medium, and its explicit effect on the macroscopic properties. In addition such an effect can be vulcanized, as obviously occurs together with external perturbation upon transition of a laminar liquid flow to turbulent.

Is it then possible to consider this fact in some manner, remaining within a phenomenological approach without a priori recourse to unavoidable averaging of the fundamental hydrodynamic equations in the region of the completed transition? Most important, using this path can we predict the region of transition from laminar to turbulent flow, using the same description in both cases? We will attempt to provide a positive answer to this question.

In view of the presence of an intrinsic fluctuation mechanism in the medium, it is proper to supplement the well-defined hydrodynamic description with probability relationships, i.e., together with the mechanical prediction that at a given time at a given point in the space $\mathbf{r}$ the liquid will have a velocity $\mathbf{v}$, we commence from the fact that the given property, having a velocity $\mathbf{v}$, has a probability distribution law: this property may manifest itself with one or the other probability in the vicinity of the point $\mathbf{r}$. The vicinity of effective appearance of the expected property may prove to be large, and is the defining characteristic of liquid behavior.

We introduce $f(\mathbf{r}'|\mathbf{v}(\mathbf{r}))$, the density of the probability that within the volume dV about the point $\mathbf{r}'$ the medium has a velocity $\mathbf{v}$, found as a solution of the Navier–Stokes equation for the point of space $\mathbf{r}$. The function defined in this manner has the dense of a conditional distribution function for the original steady-state liquid flow. The initially proposed laminar flow hypothetically remains the same for any velocity values (Reynolds numbers).

It is evident that when the flow velocity at all points of the medium is identical ($\mathbf{v}$ = const), we have the limiting case

$$f(\mathbf{r}'|\mathbf{v}(\mathbf{r})) = \delta(\mathbf{r}' - \mathbf{r}). \tag{1}$$

In this case existence of the velocity $\mathbf{v}$ at the point $\mathbf{r}$ is a certainty.

In the future, we will take a Gaussian distribution law for f. Expanding the dispersion coefficients of the distribution law in a series in velocity gradients and considering the possibility of Eq. (1), for the mean square deviation σ of the isotropic liquid, in the first approximation we obtain

$$\sigma = a(\nabla \mathbf{v})^2, \quad a = \text{const}. \tag{2}$$

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We will consider the difference $\sigma_1 - \sigma_2$ which must depend on the difference between two adjacent points of the medium $|\mathbf{r}_1 - \mathbf{r}_2|$. The relationship between them may in principle be of a different character for the determination of the laminar and turbulent liquid flows. The first case will correspond to the inequality

$$|\sigma_1 - \sigma_2| < \beta |\mathbf{r}_1 - \mathbf{r}_2|, \ \beta = \text{const}, \tag{3}$$

and the second, to the converse inequality.

The fact that both cases are possible can easily be proved with the example of a steady state Poiseuille flow. With a constant pressure gradient (x-axis directed along the tube) for any two points in a tube cross section and a single radial direction

$$|\sigma_1 - \sigma_2| = \frac{a}{4\eta^2}\left(\frac{dp}{dx}\right)^2 |r_1^2 - r_2^2|, \tag{4}$$

and Eq. (3) implies that

$$\frac{a^*}{4\eta^2}\left(\frac{dp}{dx}\right)^2 (2r_1 + H) < 1, \ H = r_2 - r_1 > 0, \ a^* = \frac{a}{\beta}. \tag{5}$$

The boundary of transition of the laminar flow into turbulent is related to transformation of Eq. (5) into an equation from which we define

$$a^* = \frac{2\eta^2}{R\left(\frac{dp}{dx}\right)_{cr}^2}, \ 2r_1 + H \leqslant 2R. \tag{6}$$

In fact, if the pressure gradient is less than the critical value, inequality (5) with consideration of Eq. (6) is satisfied for any values of r_1 and r_2. This will definitely not occur if the pressure gradient is higher than the critical value.

The fact that after disruption of Eq. (3) a qualitatively new liquid state develops is evident form general considerations: since Eq. (3) refers to any two points, including two points separated by an arbitrarily small distance, disruption of the inequality also implies "destruction" of the differential equation describing the laminar flow. The derivation of the equation becomes invalid because the derivative of the velocity loses its meaning, becoming a random quantity.

From the above there follow three consequences, which have been confirmed experimentally. The first of these is the fact that turbulence is not generated simultaneously over the entire section as the pressure gradient increases, but infailingly commences from the tube surface. It is simple to determine the thickness of the turbulent layer $R - R_0$ by using the inequality which is the converse of Eq. (5):

$$R_0 = R\left[2\frac{\left(\frac{dp}{dx}\right)_{cr}^2}{\left(\frac{dp}{dx}\right)^2} - 1\right]. \tag{7}$$

Now Eq. (5) is valid only for points of the flow core with radius R_0. The converse inequality is valid for the turbulent wall layer. Complete turbulization of the flow sets in an $dp/dx = \sqrt{2}(dp/dx)_{cr}$ when R_0 vanishes. The second consequence is closely related to the first. Since the pressure gradient is proportional to the Reynolds number, the width of the region of total transition of laminar flow into turbulent is defined by the ratio $Re/Re_{cr} = \sqrt{2}$. With increase in Re_{cr} the width of the transition region increases.

We will note in passing that for liquid flow between two fixed parallel plates the height of the wall layer with turbulent motion y is defined by an expression:

$$y = \frac{h}{2}\left[1 - \frac{\left(\frac{dp}{dx}\right)^2_{cr}}{\left(\frac{dp}{dx}\right)^2}\right], \tag{8}$$

where h is the distance between the plates. As is evident from Eq. (8), in contrast to flow in a tube, in principle turbulization sets in as $|dp/dx| \to \infty$.

From the above there also follows the third consequence – the necessary existence of a boundary layer. This is especially clear in the example of Couette flow. In the absence of such a layer Eq. (3) would never be disrupted (the left side is always equal to zero) and turbulent flow could not develop.

At the same time, it is evident that disruption of condition (3) is not sufficient for transition of laminar flow to turbulent, since it does not contain the requirement of existence of a minimum turbulence scale (Kolmogorov hypothesis). According to condition (3) the thickness of the initial turbulent wall layer can be arbitrarily small, which as will be shown below, is not in reality the case.

In the future we will consider only Poiseuille flow along a tube of radius R. We divide the radius into a number of intervals and consider the adjacent segments b_1 (from the wall) and b_2. Let v_1, the velocity at one point of the segment b_1, be a solution of the Navier–Stokes equation with the assumption of existence of a steady state boundary layer. Then v_2 is a corresponding value for a point of the segment b_2.

Disruption of condition (3) implies that the first layer of liquid (of thickness b_1) "entangles" layer b_2 in the sense that now without doubt at $\sigma_1 >> \sigma_2$ in the second internal segment there may be a velocity v_2 or v_1 (the latter, naturally, as the liquid transport velocity).

In fact, if we term the appearance of the velocity v_1 within the finite interval by dy belonging to the interval b_2 the event A, and that of the appearance of the velocity v_2, event B, then the probability of the event A + B (where A and B are comparable and independent events)

$$P(A+B) = P(A) + P(B) - P(AB)$$

where all terms on the right now become comparable to each other.

Thus, in the second layer, an alternation of laminar and turbulent liquid motion sets in (self-oscillating regime). Intense interaction of the liquid layers begins, this being a defining feature of turbulence.

We will divide the radius R into a finite, but arbitrary number of segments b_k. Making use of Eq. (2), we can construct an integral distribution function for a given velocity v, calculated for an arbitrary fixed points of the given segment. If for this point we have y_{0k-1} in the segment b_{k-1} (the y-axis directed along the radius R), in which the velocity v_{k-1} is found as a solution of the Navier–Stokes equation, then the integral distribution function $\Phi(y_{k-1})$ will be equal to:

$$\Phi = \frac{1}{\sqrt{2\pi}\sigma_{k-1}} \int_{-\infty}^{y_{k-1}} \exp\left\{-\frac{(y_{k-1} - y_{0k-1})^2}{2\sigma^2_{k-1}(y_{0k-1})}\right\} dy_{k-1}. \tag{9}$$

We construct an analogous function for the adjacent segment b_k. It is always true here that $\sigma_{k-1}(y_{0k-1}) > b_k(y_{0k})$. In light of this, their difference can be related to the fact that beginning at some value of Reynolds number the first function $\Phi(y_{k-1})$ intersects the second $\Phi(y_k)$, within the adjacent interval b_k. There occurs a "collision" of the two curves, which is an analog of collision of two adjacent layrs of a hypothetically laminar liquid flow. This occurs when

$$y_k \leqslant \frac{b_{k-1}}{2} + b_k, \tag{10}$$

if we relate y_{0k-1} to the core of the segment b_{k-1} and take $y_{0k-1} = 0$, the values of the two integral distribution functions become identical at the point $y^*_{k-1} \equiv y^*_k$:

$$\Phi(y^*_{k-1},\ \sigma_{k-1}) = \Phi(y^*_k,\ \sigma_k). \tag{11}$$

In principle Eq. (11) permits us to find the coordinate y^*, and together with Eq. (10) to determine yet one more condition, independent of Eq. (3), for the difference $\sigma_{k-1}-\sigma_k$. However, for intermediate qualitative (preliminary) estimates the search for the point of intersection of the two integral distribution curves can be replaced by a search for the point of intersection of the corresponding differential distribution curves (their right-hand branches), which in principle is no departure from the ideology under consideration. It then follows that

$$y^*_k \simeq \frac{b_{k-1}+b_k}{2}\left[1+\frac{\sigma_k}{\sigma_{k-1}}\sqrt{1+\frac{8A\sigma^2_{k-1}}{(b_{k-1}+b_k)^2}\ln\frac{\sigma_{k-1}}{\sigma_k}}\right],$$
$$A = \frac{\sigma^2_{k-1}-\sigma^2_k}{\sigma^2_{k-1}}, \tag{12}$$

and for the unknown differences $\sigma_{k-1}-\sigma_k$, using Eqs. (10) and (12), we obtain an additional condition for turbulization of the flow in the liquid layer:

$$\sigma_{k-1}-\sigma_k \geqslant \frac{\sigma_{k-1}(b_{k-1}+b_k)}{(b_{k-1}+2b_k)(\sigma_{k-1}+\sigma_k)}\left[\sigma_{k-1}+\sigma_k\sqrt{1+\frac{8A\sigma^2_{k-1}}{(b_{k-1}+b_k)^2}\ln\frac{\sigma_{k-1}}{\sigma_k}}\right]. \tag{13}$$

In contrast to the first (necessary) condition for disruption of laminar flow, Eq. (3), condition (13) does not admit an arbitrarily small turbulization layer thickness, i.e., there is in fact a finite turbulence scale. We are thus justified in terming condition (13) sufficient.

In fact, by considering two adjacent layers b_1 and b_2, it is easily proved that those layers cannot be arbitrarily small without disrupting condition (13). It is significant that the thickness of the first laminar layer at the wall b_1 has a minimum value. And it is just this layer which serves as an "external" source of disturbance for the adjacent layer b_2, the latter in turn plays the same role for the third layer, etc.

Our entire examination has been based on a well-known expansion of dynamic formalism. On the one hand, for the functions $\Phi(t)$ used here, a solution of the hydrodynamic problem (the Navier–Stokes equation) is required, while on the other, the duration of the existence of such a solution (laminar flow) is finite and is periodically replaced by the time of existence of the turbulent vortex. If the total period is taken equal to unity, then it is convenient to denote the intervals referred to above as $2\Delta t_k$, the time (probability) of existence of a transport velocity for the liquid layer in the segment b_k and $1-2\Delta t_k$, the lifetime of the initial laminar flow velocity v_k on the same interval b_k. The limit of these time intervals is defined by condition (11):

$$1-2\Delta t_k = \Phi(y^*). \tag{14}$$

Here and below Φ is twice the value of the probability integral in the interval $[0,\ y^*_{k-1}]$.

With the aid of the integral distribution functions Eq. (9) one can formally obtain two equations $y_{k-1}(t)$ and $y_k(t)$, which take on the meaning of quasikinematic motion equations. The fundamental proposition is based on the fact that interaction of the liquid layers as a cause of turbulization can be described in the language of collisions of the indicated motion equations. Therefore $2\Delta t_k$ is the period from the moment of intersection of the curves $y_{k-1}(t)$ and $y_k(t)$ to the end of the period (the lifetime of the turbulent vortex). The dynamic formalism admits a transformation operator, the eigenvalues of which are the unknown quantities Δt_k. Up to some critical value (the first critical Reynolds number) the operator functions "at idle," (there is no intersection of the curves $y_{k-1}(t)$ and $y_k(t)$ in the segment b_k) and becomes significant at $Re > Re_{cr}$. There then occurs a division of the flow velocity into a transport translational component related to the adjacent layer and a relative rotary velocity.

Equating the values of the difference $\sigma_{k-1}-\sigma_k$ from the two conditions for onset of turbulization, Eqs. (2) and (13), we achieve the possibility of determining into how many layers the turbulized liquid flow will most probably be divided. For the first two layers b_1 and b_2 (turbulence is initiated in the segment b_2 and at $b_1 + b_2 \ll R$ in view of the closeness of the values σ_1 and σ_2 one can simplify Eq. (12)) we write

$$\beta(r_1 - r_2) = \frac{\sigma_1(b_1 + b_2)}{b_1 + 2b_2}. \tag{15}$$

For Poiseuille flow

$$\sigma_1 \approx \frac{\gamma^2}{8R}(2R - b_1)^2,$$

(where γ is the ratio of the pressure gradient (head) to its critical value; σ_1 is calculated for the middle of the segment b_1: r_1 and r_2 also correspond to the middle areas of b_1 and b_2 ($r_1-r_2 = (b_1 + b_2)/2$). With the aid of Eq. (15) it is simple to obtain the condition

$$b_1 + b_2 \leqslant \frac{\gamma^2}{2\beta} R. \tag{16}$$

For the case of formation of three layers, instead of the two considered above, we obtain the more severe condition which begins to be satisfied at higher values of γ (the pressure head). A finer division of one and the same portion of the flow into individual layers will occur with increase in γ, i.e., Renolds number. This occurs discontinuously, which corresponds to known model representations of chaotization of the liquid flow (see [5, 6]). In addition it is now possible to estimate the lifetime (probability) of one or the other turbulence scale, and their interchangability. A multifrequency turbulence mechanism develops. With increase in Reynolds number the probability of appearance of large scales vanishes.

In conclusion, we will turn to the simplest example of existence of two layers $b_1 = R/3$ and $b_2 = 2R/3$. Then, according to Eq. (14):

$$1 - 2\Delta t_2 = \Phi(\gamma). \tag{17}$$

We define the velocities v_1 and v_2 from the condition of conservation of the hypothetically laminar flow $\left(v_1 = \frac{5R^2}{72\eta}\left|\frac{dp}{dx}\right| \text{ and } v_2 = \frac{7R^2}{36\eta}\left|\frac{dp}{dx}\right|\right)$. The mean flow velocity of the transport motion of the turbulent liquid U is then definable in terms of the probabilities $2\Delta t_2$ and $1 - 2\Delta t_2$:

$$U = \frac{5}{9}v_1 + \frac{4}{9}[v_2\Phi(\gamma) + v_1(1 - \Phi(\gamma))]. \tag{18}$$

For the tube resistance coefficient we obtain the final expression

$$\frac{\lambda_\gamma}{\lambda_{\gamma=1}} = \frac{81}{\gamma(5 + 4\Phi(\gamma))^2}. \tag{19}$$

For $\gamma \leq 1$ $\Phi \cong 1$ and $\lambda \sim 1/\gamma$ (laminar flow), but with further growth in γ (pressure gradient), the λ values begin to increase and the curve passes through a maximum in the interval $1 < \gamma < \sqrt{2}$. The function (19) reproduces the characteristic features of the known experimental curves of resistance coefficient vs Reynolds number.

The probability integral Φ itself is significant for the transition region, after which its role fades rapidly. This means that the lifetime of a periodically occurring laminar flow $1 - 2\Delta t \to 0$. The system becomes constantly turbulized, and essentially, more organized, as is confirmed by Klimontovich's S-theorem on decrease of entropy upon transition from laminar to turbulent flow [7].

With increase in γ, according to Eq. (18), the velocity $U \to v_1$ and the velocity profile collapses severely. Now the expression for the dispersion coefficient, Eq. (2), should be supplemented by the following term of the "expansion" in the velocity gradient, which increases a great deal.

NOTATION

$\mathbf{r}$, radius vector; x, y, coordinates; $\mathbf{v}$, velocity; δ, Dirac function; f, probability density; Φ, integral distribution function; σ, mean square deviation; ∇, gradient operator; η, shear viscosity coefficient; p, pressure; R, tube radius; t, time; Re, Reynolds number; γ, ratio of Reynolds number to critical value; U, mean flow velocity; λ, tube resistance coefficient.

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DEVELOPMENT AND MOVEMENT OF VORTEX STRUCTURES NEAR THE SURFACE OF A SOLID WALL

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The dynamics of large-scale vortices near the surface of a plate is modeled on the basis of numerical integration of nonsteady two-dimensional Navier–Stokes equations.

Large-scale vortex structures formed near the surface of a solid wall have a significant influence on momentum transfer and can be used as a means of controlling flow and heat transfer in the boundary layer [1, 2]. In this light, a study we made of the development and movement of vortices near the surface of a solid wall is of more than just theoretical interest. It also makes it possible to at least qualitatively evaluate the effect of vortices on such important flow characteristics as wall friction.

The object of our study was a semiinfinite flat horizontal plate. The vortices were generated by a circular cylinder positioned symmetrically relative to the plate at its front edge. Provision was made for alternate rotation of the cylinder in the clockwise and counterclockwise directions at a specified velocity. The method of study was based on numerical integration of the complete two-dimensional Navier–Stokes equations. Before analyzing the results, we will describe features of the numerical method – which was designed for modeling nonsteady flows. We will also first demonstrate its use in an example involving the solution of a test problem concerning the development of a wake behind an isolated circular cylinder.

1. Numerical Procedure. In constructing the numerical method, we use a difference grid and calculate the velocity components at nodes located a half-step from the nodes where pressure is recorded. Figure 1 shows an element of the grid and a characteristic control volume surrounding the node P. Integration over the control volume reduces the system of equations of continuity and momentum change to the form:

$$I = 0, \tag{1}$$

$$(\partial\Phi/\partial t)\,\mathrm{vol} + F = 0, \tag{2}$$

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